Research support project of the University of Rijeka - uniri-prirod-18-118-1257.

Analysis of mathematical models of fluid mechanics and technical systems using data-driven algorithms for Koopman operator

Team members: Nelida Črnjarić-Žic, Senka Maćešić, Ivan Dražić, Loredana Simčić, Angela Bašić-Šiško, Igor Mezić (UCSB), Stefan Ivić

University of Rijeka, Faculty of Engineering, Vukovarska 58, 51000 Rijeka, Croatia

Themes and results

- Micropolar flow between parallel plates - the development of the model and the analysis of the solution
- 1-D flow of a p-th power viscous micropolar and heat-conducting fluid - the development od the model, mathematical and numerical analysis
- The extension of Koopman models to stochastic and nonautonomous systems and their analysis
- The improvements of the theory and the applications of data driven algorithms and their application to the dynamical systems
- Prediction of the dynamical systems using data driven algorithms

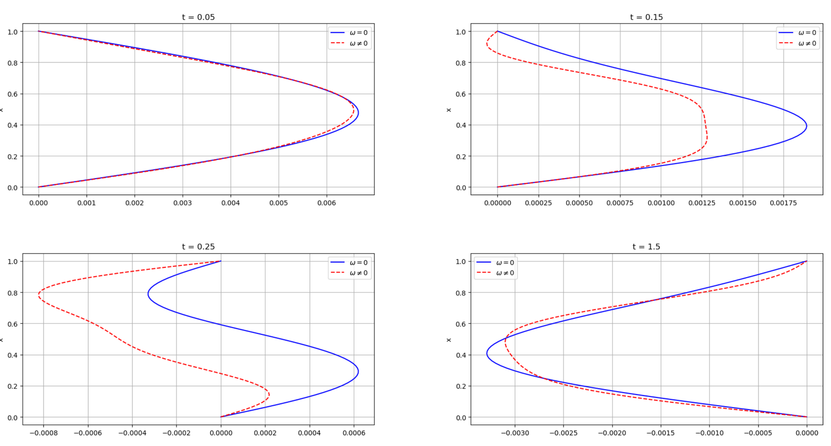

Fig. 1: Displacement from the initial position - comparison of classical and micropolar fluid models.

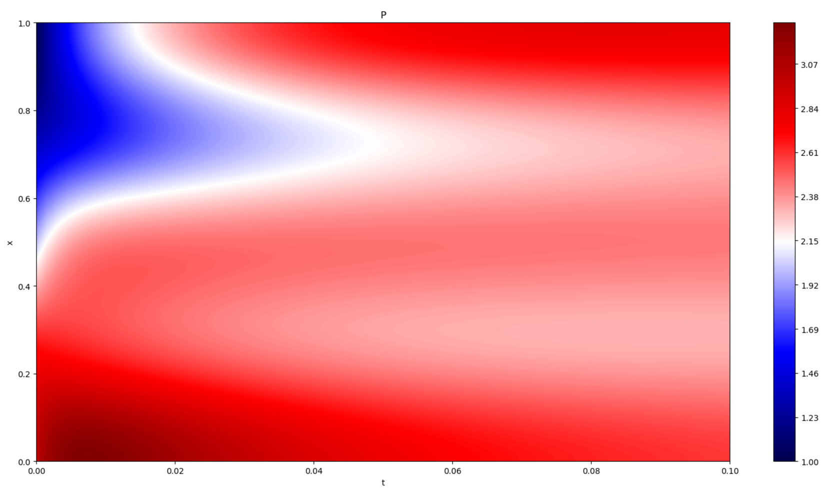

Fig. 2: Pressure $P(x,t) = R\rho(x,t)^p\theta(x,t)$.

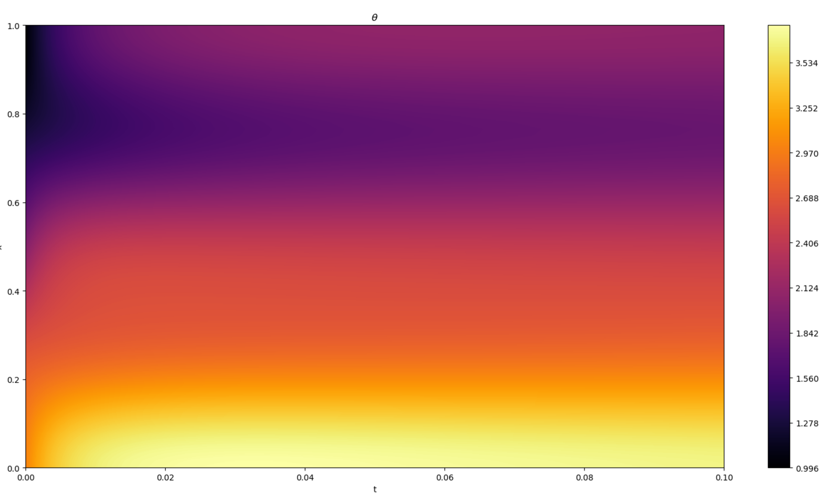

Fig. 3: Absolute temperature $\theta(x,t)$.

1-D flow of a p-th power viscous micropolar and heat-conducting fluid

In micro and nanosciences, classical fluid models can rarely be adequately considered, since micro-phenomena, which cannot be covered using classical models, increasingly come to the fore. Therefore, the models which can describe the phenomena at the micro level are increasingly analyzed. One example of such a model is the model of the micropolar continuum, which is the subject of this research. The application of the micropolar fluid model has proven important in a number of areas such as biology and medicine for modelling of biological flows, and mechanical engineering for example for modelling the behaviour of lubricants.

We consider a one-dimensional model of viscous and heat-conducting micropolar real gas flow through the channel with solid and thermally insulated walls. Real gas is characterized by the generalized equation of state

$$P = R\rho^p\theta,$$

where P denotes pressure, $R > 0$ is a constant, and $p \geq 1$ is a constant called pressure exponent.
The governing initial-boundary value problem in Lagrangian description is given as

$$\begin{aligned}
\partial_t\rho &= -\frac{1}{L}\rho^2\partial_x v,\\
\partial_t v &= -\frac{R}{L}\partial_x(\rho^p\theta) + \frac{\lambda+2\mu}{L^2}\partial_x(\rho\partial_x v),\\
j_I\partial_t\omega &= \frac{c_0+2c_d}{L^2}\partial_x(\rho\partial_x\omega) - 4\mu_r\frac{\omega}{\rho},\\
c_v\partial_t\theta &= \frac{\kappa}{L^2}\partial_x(\rho\partial_x\theta) - \frac{R}{L}\rho^p\theta\,\partial_x v + \frac{\lambda+2\mu}{L^2}\rho(\partial_x v)^2 + \frac{c_0+2c_d}{L^2}\rho(\partial_x\omega)^2 + 4\mu_r\frac{\omega^2}{\rho},
\end{aligned}$$

for $(x,t) \in]0,1[\times]0,T[$, with initial conditions

$$\rho(x,0) = \rho_0(x), \quad v(x,0) = v_0(x), \quad \omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x),$$

for $x \in [0,1]$, and homogeneous boundary conditions

$$v(0,t) = v(1,t) = 0, \quad \omega(0,t) = \omega(1,t) = 0, \quad \partial_x\theta(0,t) = \partial_x\theta(1,t) = 0,$$

for $t \in [0,T]$, whereby $\rho = \rho(x,t)$, $v = v(x,t)$, $\omega = \omega(x,t)$, and $\theta = \theta(x,t)$ denote mass density, velocity, microrotation, and absolute temperature, respectively. L is a positive constant, λ and μ are coefficients of viscosity, μ_r, c_0, c_d are coefficients of microviscosity, j_I is microinertia density, c_v is specific heat at constant volume, and κ is heat conductivity coefficient.

It is shown that the described system has a unique solution locally in time and that it can be extended to a time interval of arbitrary length by the principle of extension. Using the Faedo-Galerkin method, a numerical scheme is constructed for the described system. The obtained numerical solutions are shown in Figures 1-3.

A Koopman Operator-Based Prediction Algorithm and its Application

The black-swan-type dynamics occurs when an underlying dynamical system becomes coupled to, or decoupled from, another one. Here we explore the problem of prediction in systems that exhibit such behavior. The mathematical theory and algorithms we use are based on an operator-theoretic approach in which the dynamics of the system are embedded into an infinite-dimensional space. We show that the framework correctly identifies a black swan event. Moreover, we show that the algorithms we developed enabled a successful prediction of the flu season, and prediction in other complex dynamics datasets.
The Koopman operator family $\mathcal{K}^t$, acts on observables f by composition $\mathcal{K}^t f(\mathbf{x}) = f(\mathbf{x}(t))$. It is a linearization tool: $\mathcal{K}^t$ is a linear operator that allows studying the nonlinear dynamics by examining its action on a linear space $\mathcal{F}$ of observables. In data analysis, we use discrete time steps t_i and the discrete sequence $\mathbf{z}_i \approx \mathbf{x}(t_i)$, generated as numerical software output, is then a discrete dynamical system $\mathbf{z}_{i+1} = \mathbf{T}(\mathbf{z}_i)$, for which the Koopman operator reads $\mathcal{K}f = f \circ \mathbf{T}$.
The key of the spectral analysis of the dynamical system is a representation of a vector valued observable $\mathbf{f} = (f_1,\ldots,f_d)^T$ as a linear combination of the eigenfunctions ψ_j of $\mathcal{K}$. In a subspace spanned by eigenfunctions each observable f_i can be written as $f_i(\mathbf{z}) \approx \sum_{j=1}^{\infty}\psi_j(\mathbf{z})(\mathbf{v}_j)_i$ and thus

$$\mathbf{f}(\mathbf{z}) = \begin{pmatrix} f_1(\mathbf{z})\\ \vdots \\ f_d(\mathbf{z})\end{pmatrix} \approx \sum_{j=1}^{\infty}\psi_j(\mathbf{z})\mathbf{v}_j, \text{ where } \mathbf{v}_j = \begin{pmatrix} (\mathbf{v}_j)_1\\ \vdots \\ (\mathbf{v}_j)_d\end{pmatrix}, \tag{1}$$

then, since $\mathcal{K}\psi_j = \lambda_j\psi_j$, we can envisage the values of the observable $\mathbf{f}$ at the *future* states $\mathbf{T}(\mathbf{z}), \mathbf{T}^2(\mathbf{z}),\ldots$ by

$$(\mathcal{K}^k\mathbf{f})(\mathbf{z}) \stackrel{\text{def}}{=} \mathbf{f}(\mathbf{T}^k(\mathbf{z})) \approx \sum_{j=1}^{\infty}\lambda_j^k\psi_j(\mathbf{z})\mathbf{v}_j, \quad k = 1,2,\ldots \tag{2}$$

The decomposition (1) is called the *Koopman Mode Decomposition* (KMD); the scalars λ_j are the Koopman eigenvalues, and the $\mathbf{v}_j$'s are the Koopman modes. Its numerical approximation can be computed based on the supplied data pairs $(\mathbf{f}(\mathbf{z}_i), \mathbf{f}(\mathbf{T}(\mathbf{z}_i)))$, $i = 0,\ldots,M$, using e.g. the *Dynamic Mode Decomposition*.
In Figure 4 we show the Ritz values obtained by using DMD RRR algorithm, applied on influenza data. In Figure 5 we show the prediction using the global prediction algorithm. The active window (shadowed rectangle) is July 2004 – July 2010, and the dynamics is predicted for 104 weeks ahead. The global prediction fails due to the Black Swan data in the learning window. The global prediction algorithm recovers after the retouching the Black Swan event data, which allows for using big learning window. The retouching technique that repairs the distorted training data restores the intrinsic dynamics over the entire training window. The distribution of the relevant eigenvalues becomes more consistent before and after retouching of data.

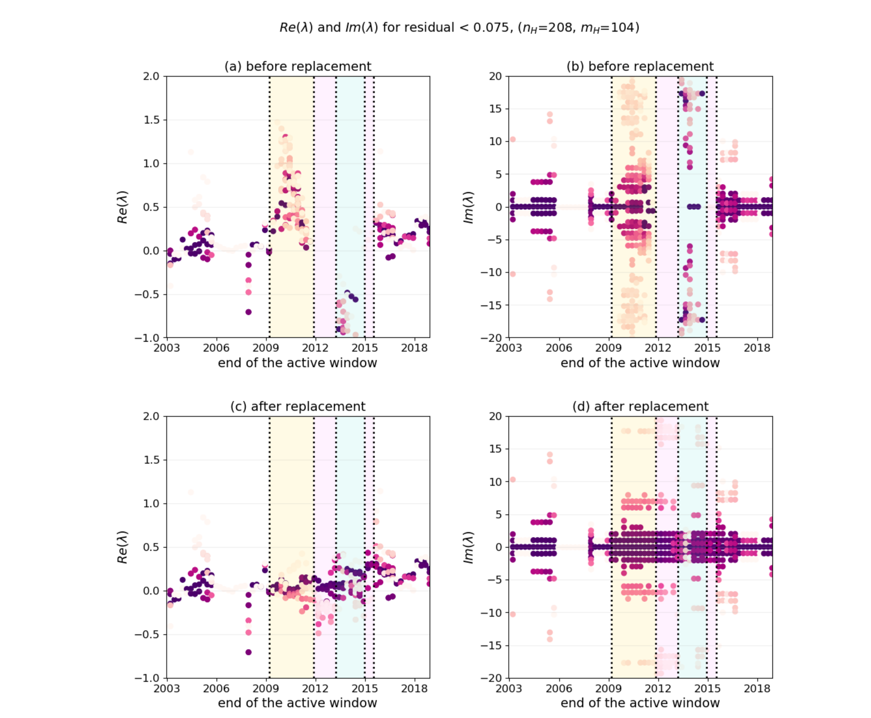

Fig. 4: The real and imaginary parts of Ritz values with residuals bellow $\eta_r = 0.075$ for sliding active windows.

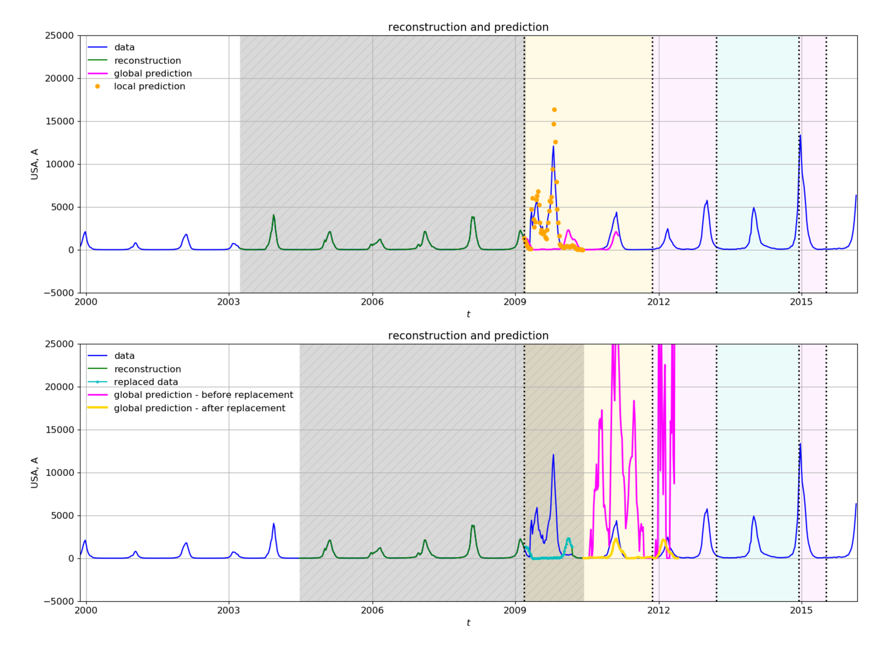

Fig. 5: Global prediction with retouching the Black Swan event.